Calculating two-point resistances in distance-regular resistor networks

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Abstract

An algorithm for the calculation of the resistance between two arbitrary nodes in an arbitrary distance-regular resistor network is provided, where the calculation is based on stratification introduced in [1] and Stieltjes transform of the spectral distribution (Stieltjes function) associated with the network. It is shown that the resistances between a node α and all nodes β belonging to the same stratum with respect to the α ($R_{\alpha\beta^{(i)}}$, β belonging to the i-th stratum with respect to the α) are the same. Also, the analytical formulas for two-point resistances $R_{\alpha\beta^{(i)}}, i = 1, 2, 3$ are given in terms of the the size of the network and corresponding intersection numbers. In particular, the two-point resistances in a strongly regular network are given in terms of the its parameters $(v, \kappa, \lambda, \mu)$. Moreover, the lower and upper bounds for two-point resistances in strongly regular networks are discussed.

Keywords:two-point resistance, association scheme, distance-regular networks, Stieltjes function

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1 Introduction

A classic problem in electric circuit theory studied by numerous authors over many years is the computation of the resistance between two nodes in a resistor network (see, e.g., [2]). Besides being a central problem in electric circuit theory, the computation of resistances is also relevant to a wide range of problems ranging from random walks (see [3]), the theory of harmonic functions [4], to lattice Greens functions [5, 6, 7, 8, 9]. The connection with these problems originates from the fact that electrical potentials on a grid are governed by the same difference equations as those occurring in the other problems. For this reason, the resistance problem is often studied from the point of view of solving the difference equations, which is most conveniently carried out for infinite networks. In the case of Greens function approach, for example, past efforts [2], [23] have been focused mainly on infinite lattices. Little attention has been paid to finite networks, even though the latter are those occurring in real life. In this paper, we take up this problem and present a general formulation for computing two-point resistances in finite networks. Particularly, we show that known results for infinite networks are recovered by taking the infinite-size limit.

The study of electric networks was formulated by Kirchhoff [11] more than 150 years ago as an instance of a linear analysis. Our starting point is along the same line by considering the Laplacian matrix associated with a network. The Laplacian is a matrix whose off-diagonal entries are the conductances connecting pairs of nodes. Just as in graph theory where everything about a graph is described by its adjacency matrix (whose elements is 1 if two vertices are connected and 0 otherwise), everything about an electric network is described by its Laplacian. The author of [12], has been derived an expression for the two-point resistance between two arbitrary nodes α and β of a regular network in terms of the matrix entries $L^{-1}_{\alpha\alpha}$, $L^{-1}_{\beta\beta}$ and $L^{-1}_{\alpha\beta}$, where L^{-1} is the pseudo inverse of the Laplacian matrix. Here in this work, based on stratification introduced in [1] and spectral analysis method, we introduce an procedure for

calculating two-point resistances in distance-regular resistor networks in terms of the Stieltjes function $G_\mu(x)$ associated with the adjacency matrix of the network and its derivatives. Although, we discuss the case of distance-regular networks, but also the method can be used for any arbitrary regular network. It should be noticed that, in this way, the two-point resistances are calculated straightforwardly without any need to know the spectrum of the network. Also, it is shown that the resistances between a node α and all nodes β belonging to the same stratum with respect to the α ($R_{\alpha\beta^{(i)}}$, β belonging to the i-th stratum with respect to the α) are the same. We give the analytical formulas for two-point resistances $R_{\alpha\beta^{(i)}}, i = 1, 2, 3$ in terms of the network's characteristics such as the size of the network and its intersection array. In particular, the two-point resistances in a strongly regular network are given in terms of the network's parameters $(v, \kappa, \lambda, \mu)$. Moreover, we discuss the lower and upper bounds for two-point resistances in strongly regular networks. From the fact that, the two-point resistances on a network depend on the corresponding Stieltjes function $G_\mu(x)$ and that $G_\mu(x)$ is written as a continued fraction, the two-point resistances on an infinite-size network can be approximated with those of the corresponding finite-size networks.

The organization of the paper is as follows. In section 2, we give some preliminaries such as association schemes, distance-regular networks, stratification of these networks and Stieltjes function associated with the network. In section 3, two-point resistances in distance-regular networks are given in terms of the Stieltjes function and its derivatives. Also, the resistances $R_{\alpha\beta^{(i)}}$, for $i = 1, 2, 3$ are given in terms of the network's intersection array. In particular, two-point resistances in a strongly regular network are given in terms of the network's parameters, also lower and upper bounds for the two-point resistances in these networks are discussed. Section 4 is devoted to calculating two-point resistances $R_{\alpha\beta^{(i)}}$ for $i = 1, 2, 3$ in some important examples of distance-regular networks, such as complete network, strongly regular networks (distance-regular networks with diameter 2), e.g. Petersen and normal subgroup scheme networks [1], d-cube (d dimensional hypercube) and Johnson networks. The paper is ended with a

brief conclusion and an appendix containing a table for two-point resistances $R_{\alpha\beta^{(i)}}$, $i = 1, 2, 3$ of some important distance-regular resistor networks with size less than 70.

2 Preliminaries

In this section we give some preliminaries such as definitions related to association schemes, corresponding stratification, distance-regular networks and Stieltjes function associated with the network.

2.1 Association schemes

First we recall the definition of association schemes. The reader is referred to Ref.[13], for further information on association schemes.

Definition 2.1 (Symmetric association schemes). Let V be a set of vertices, and let $R_i(i = 0, 1, ..., d)$ be nonempty relations on V (i.e., subset of $V \times V$). Let the following conditions (1), (2), (3) and (4) be satisfied. Then, the relations $\{R_i\}_{0\leq i\leq d}$ on $V \times V$ satisfying the following conditions

(1) $\{R_i\}_{0\leq i\leq d}$ is a partition of $V \times V$

(2) $R_0 = \{(\alpha, \alpha) : \alpha \in V\}$

(3) $R_i = R_i^t$ for $0 \leq i \leq d$, where $R_i^t = \{(\beta, \alpha) : (\alpha, \beta) \in R_i\}$

(4) For $(\alpha, \beta) \in R_k$, the number $p_{i,j}^k =\mid \{\gamma \in X : (\alpha, \beta) \in R_i \;\; and \;\; (\gamma, \beta) \in R_j\} \mid$ does not depend on (α, β) but only on i, j and k,

define a symmetric association scheme of class d on V which is denoted by $Y = (V, \{R_i\}_{0\leq i\leq d})$. Furthermore, if we have $p_{ij}^k = p_{ji}^k$ for all $i, j, k = 0, 2, ..., d$, then Y is called commutative.

The intersection number p_{ij}^k can be interpreted as the number of vertices which have relation i and j with vertices α and β, respectively provided that $(\alpha, \beta) \in R_k$, and it is the same for all element of relation R_k. For all integers i $(0 \leq i \leq d)$, set $k_i = p_{ii}^0$ and note that $k_i \neq 0$,

since R_i is non-empty. We refer to k_i as the i-th valency of Y.

Let $Y = (X, \{R_i\}_{0 \leq i \leq d})$ be a commutative symmetric association scheme of class d, then the matrices $A_0, A_1, ..., A_d$ defined by

$$(A_i)_{\alpha,\beta} = \begin{cases} 1 & \text{if } (\alpha, \beta) \in R_i \\ 0 & \text{otherwise} \end{cases}, \tag{2-1}$$

are adjacency matrices of Y such that

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k. \tag{2-2}$$

From (2-2), it is seen that the adjacency matrices $A_0, A_1, ..., A_d$ form a basis for a commutative algebra A known as the Bose-Mesner algebra of Y.

Definition 2.2 (P-polynomial property) $Y = (X, \{R_i\}_{0 \leq i \leq d})$ is said to be P-polynomial (with respect to the ordering $R_0, ..., R_d$ of the associate classes) whenever for all integers h, i, j $(0 \leq h, i, j \leq d)$,

$$p_{ij}^h = 0 \quad \text{if one of} \quad h, i, j \quad \text{is greater than the sum of the other two,}$$

$$p_{ij}^h \neq 0 \quad \text{if one of} \quad h, i, j \quad \text{equals the sum of the other two.} \tag{2-3}$$

It is shown in [14] that in the case of P-polynomial schemes, $A_i = P_i(A)$ $(0 \leq i \leq d)$, where P_i is a polynomial with real coefficients and degree exactly i. In particular, A multiplicatively generates the Bose-Mesner algebra.

Finally the underlying graph of an association scheme $\Gamma = (V, R_1)$ is an undirected connected graph, where the set V and R_1 consist of its vertices and edges, respectively. Obviously replacing R_1 with one of other relation such as R_i, for $i \neq 0, 1$ will also give us an underlying graph $\Gamma = (V, R_i)$ (not necessarily a connected graph) with the same set of vertices but a new set of edges R_i.

2.2 Stratifications

For a given vertex $\alpha \in V$, let $R_i(\alpha) := \{\beta \in V : (\alpha,\beta) \in R_i\}$ denote the set of all vertices having the relation R_i with α. Then, the vertex set V can be written as disjoint union of $R_i(\alpha)$ for $i = 0, 1, 2, ..., d$, i.e.,

$$V = \bigcup_{i=0}^{d} R_i(\alpha). \tag{2-4}$$

We fix a point $o \in V$ as an origin of a distance regular graph (the underlying graph of an association scheme), called reference vertex. Then, the relation (2-4) stratifies the underlying graph into a disjoint union of associate classes $R_i(o)$. With each associate class $R_i(o)$ we associate a unit vector in $l^2(V)$ defined by

$$|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{\alpha \in R_i(o)} |\alpha\rangle, \tag{2-5}$$

where, $|\alpha\rangle$ denotes the eigenket of α-th vertex at the associate class $R_i(o)$ and $\kappa_i = |R_i(o)|$ is called the i-th valency of the graph (i.e., $\kappa_i = p^0_{i,i}$). The closed subspace of $l^2(V)$ spanned by $\{|\phi_i\rangle\}$ is denoted by $\Lambda(G)$. Since $\{|\phi_i\rangle\}$ becomes a complete orthonormal basis of $\Lambda(G)$, we often write

$$\Lambda(G) = \sum_i \oplus \mathbf{C}|\phi_i\rangle. \tag{2-6}$$

Let A_i be the adjacency matrix of a graph $\Gamma = (V, R)$ for reference state $|\phi_0\rangle$ ($|\phi_0\rangle = |o\rangle$, with $o \in V$ as reference vertex), we have

$$A_i|\phi_0\rangle = \sum_{\beta \in R_i(o)} |\beta\rangle. \tag{2-7}$$

Then by using unit vectors $|\phi_i\rangle$ (2-5) and (2-7), we have

$$A_i|\phi_0\rangle = \sqrt{\kappa_i}|\phi_i\rangle. \tag{2-8}$$

2.3 Distance-regular graphs

Here in this section we consider some set of important graphs called distance regular graphs, where the relations are based on distance function defined as follows: A finite sequence

$\alpha_0, \alpha_1, ..., \alpha_n \in V$ is called a walk of length n (or of n steps) if $\alpha_{k-1} \sim \alpha_k$ for all $k = 1, 2, ..., n$. For $\alpha \neq \beta$ let $\partial(\alpha, \beta)$ be the length of the shortest walk connecting α and β, therefore $\partial(\alpha, \beta)$ gives the distance between vertices α and β hence it is called the distance function and we have $\partial(\alpha, \alpha) = 0$ for all $\alpha \in V$ and $\partial(\alpha, \beta) = 1$ if and only if $\alpha \sim \beta$. Therefore, the distance regular graphs become metric spaces with the distance function ∂.

An undirected connected graph $\Gamma = (V, R_1)$ is called distance regular graph if it is the underlying graph of a P-polynomial association scheme with relations defined as: $(\alpha, \beta) \in R_i$ if and only if $\partial(\alpha, \beta) = i$, for $i = 0, 1, ..., d$, where $d :=\max\{\partial(\alpha, \beta) : \alpha, \beta \in V\}$ is called the diameter of the graph. Usually in distance regular graphs, the relations R_i are denoted by Γ_i.

Now, in any connected graph, for every $\beta \in R_i(\alpha)$ we have

$$R_1(\beta) \subseteq R_{i-1}(\alpha) \cup R_i(\alpha) \cup R_{i+1}(\alpha). \tag{2-9}$$

Hence in a distance regular graph, $p^i_{j1} = 0$ (for $i \neq 0$, j is not $\{i-1, i, i+1\}$). The intersection numbers of the graph are defined as

$$a_i = p^i_{i1}, \quad b_i = p^i_{i+1,1}, \quad c_i = p^i_{i-1,1} \quad . \tag{2-10}$$

The intersection numbers (2-10) and the valencies κ_i satisfy the following obvious conditions

$$a_i + b_i + c_i = \kappa, \quad \kappa_{i-1}b_{i-1} = \kappa_i c_i, \quad i = 1, ..., d,$$

$$\kappa_0 = c_1 = 1, \quad b_0 = \kappa_1 = \kappa, \quad (c_0 = b_d = 0). \tag{2-11}$$

Thus all parameters of the graph can be obtained from the intersection array $\{b_0, ..., b_{d-1}; c_1, ..., c_d\}$.

It should be noticed that, for distance-regular graphs, the unit vectors $|\phi_i\rangle$ for $i = 0, 1, ..., d$ defined as in (2-5), satisfy the following three-term recursion relations

$$A|\phi_i\rangle = \beta_{i+1}|\phi_{i+1}\rangle + \alpha_i|\phi_i\rangle + \beta_i|\phi_{i-1}\rangle, \tag{2-12}$$

where, the coefficients α_i and β_i are defined as

$$\alpha_k = \kappa - b_k - c_k, \qquad \omega_k \equiv \beta^2_k = b_{k-1}c_k, \quad k = 1, ..., d, \tag{2-13}$$

i.e., in the basis of unit vectors $\{|\phi_i\rangle, i = 0, 1, ..., d\}$, the adjacency matrix A is projected to the following symmetric tridiagonal form:

$$A = \begin{pmatrix} \alpha_0 & \beta_1 & 0 & ... & ... & 0 \\ \beta_1 & \alpha_1 & \beta_2 & 0 & ... & 0 \\ 0 & \beta_2 & \alpha_3 & \beta_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \beta_{d-1} & \alpha_{d-1} & \beta_d \\ 0 & ... & 0 & 0 & \beta_d & \alpha_d \end{pmatrix}. \tag{2-14}$$

In Ref. [20], it has be shown that, the coefficients α_i and β_i can be also obtained easily by using the Lanczos iteration algorithm. Hereafter, we will refer to the parameters α_i and ω_i in (2-13) as QD (Quantum decomposition) parameters.

One should notice that, in distance regular graphs stratification is reference state independent, namely we can choose every vertex as a reference state, while the stratification of more general graphs may be reference dependent.

2.4 Stieltjes function associated with the network

It is well known that, for any pair $(A, |\phi_0\rangle)$ of a matrix A and a vector $|\phi_0\rangle$, it can be assigned a measure μ as follows

$$\mu(x) = \langle\phi_0|E(x)|\phi_0\rangle, \tag{2-15}$$

where $E(x) = \sum_i |u_i\rangle\langle u_i|$ is the operator of projection onto the eigenspace of A corresponding to eigenvalue x, i.e.,

$$A = \int xE(x)dx. \tag{2-16}$$

It is easy to see that, for any polynomial $P(A)$ we have

$$P(A) = \int P(x)E(x)dx, \tag{2-17}$$

where for discrete spectrum the above integrals are replaced by summation. Therefore, using the relations (2-15) and (2-17), the expectation value of powers of adjacency matrix A over starting site $|\phi_0\rangle$ can be written as

$$\langle\phi_0|A^m|\phi_0\rangle = \int_R x^m \mu(dx), \qquad m = 0, 1, 2, ... \tag{2-18}$$

The existence of a spectral distribution satisfying (2-18) is a consequence of Hamburgers theorem, see e.g., Shohat and Tamarkin [[15], Theorem 1.2].

Obviously relation (2-18) implies an isomorphism from the Hilbert space of the stratification onto the closed linear span of the orthogonal polynomials with respect to the measure μ. It is well known that [20] for distance-regular graphs we have

$$|\phi_i\rangle = P_i(A)|\phi_0\rangle, \tag{2-19}$$

where P_i is a polynomial with real coefficients and degree i. Now, substituting (2-19) in (2-12), we get three term recursion relations between polynomials $P_j(A)$, which leads to the following three term recursion between polynomials $P_j(x)$

$$xP_k(x) = \beta_{k+1}P_{k+1}(x) + \alpha_k P_k(x) + \beta_k P_{k-1}(x) \tag{2-20}$$

for $k = 0, ..., d-1$, with $P_0(x) = 1$.
Multiplying by $\beta_1...\beta_k$ we obtain

$$\beta_1...\beta_k x P_k(x) = \beta_1...\beta_{k+1}P_{k+1}(x) + \alpha_k\beta_1...\beta_k P_k(x) + \beta_k^2.\beta_1...\beta_{k-1}P_{k-1}(x). \tag{2-21}$$

By rescaling P_k as $Q_k = \beta_1...\beta_k P_k$, the spectral distribution μ under question is characterized by the property of orthonormal polynomials $\{Q_k\}$ defined recurrently by

$$Q_0(x) = 1, \qquad Q_1(x) = x,$$

$$xQ_k(x) = Q_{k+1}(x) + \alpha_k Q_k(x) + \beta_k^2 Q_{k-1}(x), \quad k \geq 1. \tag{2-22}$$

If such a spectral distribution is unique, the spectral distribution μ is determined by the identity

$$G_\mu(x) = \int_R \frac{\mu(dy)}{x-y} = \frac{1}{z-\alpha_0-\frac{\beta_1^2}{z-\alpha_1-\frac{\beta_2^2}{z-\alpha_2-\frac{\beta_3^2}{z-\alpha_3-\cdots}}}} = \frac{Q_{d-1}^{(1)}(x)}{Q_d(x)} = \sum_{l=0}^{d-1} \frac{A_l}{x-x_l}, \tag{2-23}$$

where, x_l are the roots of polynomial $Q_d(x)$. $G_\mu(z)$ is called the Stieltjes/Hilbert transform of spectral distribution μ or Stieltjes function and polynomials $\{Q_k^{(1)}\}$ are defined recurrently as

$$Q_0^{(1)}(x) = 1, \qquad Q_1^{(1)}(x) = x - \alpha_1,$$

$$xQ_k^{(1)}(x) = Q_{k+1}^{(1)}(x) + \alpha_{k+1}Q_k^{(1)}(x) + \beta_{k+1}^2 Q_{k-1}^{(1)}(x), \quad k \geq 1, \tag{2-24}$$

respectively. The coefficients A_l appearing in (2-23) are calculated as

$$A_l = \lim_{z \to x_l}(z - x_l)G_\mu(z). \tag{2-25}$$

(for more details see Refs.[15, 16, 17, 18].)

3 Two-point resistances in regular resistor networks

A classic problem in electric circuit theory studied by numerous authors over many years, is the computation of the resistance between two nodes in a resistor network (see, e.g., [2]). The results obtained in this section show that, there is a close connection between the techniques introduced in section 2 such as Hilbert space of the stratification and the Stieltjes function and electrical concept of resistance between two arbitrary nodes of regular networks and these techniques can be employed for calculating two-point resistances.

For a given regular graph Γ with n vertices and adjacency matrix A, let $r_{ij} = r_{ji}$ be the resistance of the resistor connecting vertices i and j. Hence, the conductance is $c_{ij} = r_{ij}^{-1} = c_{ji}$ so that $c_{ij} = 0$ if there is no resistor connecting i and j. Denote the electric potential at the i-th vertex by V_i and the net current flowing into the network at the i-th vertex by I_i (which

is zero if the i-th vertex is not connected to the external world). Since there exist no sinks or sources of current including the external world, we have the constraint $\sum_{i=1}^{n} I_i = 0$. The Kirchhoff law states

$$\sum_{j=1,j\neq i}^{n} c_{ij}(V_i - V_j) = I_i, \quad i = 1, 2, ..., n. \tag{3-26}$$

Explicitly, Eq.(3-26) reads

$$L\vec{V} = \vec{I}, \tag{3-27}$$

where, $\vec{V}$ and $\vec{I}$ are n-vectors whose components are V_i and I_i, respectively and

$$L = \sum_i c_i |i\rangle\langle i| - \sum_{i,j} c_{ij} |i\rangle\langle j| \tag{3-28}$$

is the Laplacian of the graph Γ with

$$c_i \equiv \sum_{j=1,j\neq i}^{n} c_{ij}, \tag{3-29}$$

for each vertex α. Hereafter, we will assume that all nonzero resistances are equal to r, then the off-diagonal elements of $-L$ are precisely those of $\frac{1}{r}A$, i.e.,

$$L = \frac{1}{r}(\kappa I - A), \tag{3-30}$$

with $\kappa = deg(\alpha)$, for each vertex α. It should be noticed that, L has eigenvector $(1, 1, ..., 1)^t$ with eigenvalue 0. Therefore, L is not invertible and so we define the psudo-inverse of L as

$$L^{-1} = \sum_{i,\lambda_i\neq 0} \lambda_i^{-1} E_i, \tag{3-31}$$

where, E_i is the operator of projection onto the eigenspace of L^{-1} corresponding to eigenvalue λ_i. Following the result of [12] and that L^{-1} is a real matrix, the resistance between vertices α and β is given by

$$R_{\alpha\beta} = \langle\alpha|L^{-1}|\alpha\rangle - 2\langle\alpha|L^{-1}|\beta\rangle + \langle\beta|L^{-1}|\beta\rangle. \tag{3-32}$$

In this paper, we will consider distance-regular graphs as resistor networks. Then, the diagonal entries of L^{-1} are independent of the vertex, i.e., $L^{-1}_{\alpha\alpha} = L^{-1}_{\beta\beta}$ for all $\alpha, \beta \in V$. Therefore,

from the relation (3-32), we can obtain the two-point resistance between two arbitrary nodes α and β as follows is written as

$$R_{\alpha\beta} = 2(L^{-1}_{\alpha\alpha} - L^{-1}_{\alpha\beta}). \tag{3-33}$$

Now, let α and β be two arbitrary nodes of the network such that β belongs to the l-th stratum with respect to α, i.e., $\beta \in R_l(\alpha)$ (we choose one of the nodes, here α, as reference node). Then, for calculating the matrix entries $L^{-1}_{\alpha\alpha}$ and $L^{-1}_{\beta\alpha}$ in (3-32), we use the Stieltjes function to obtain

$$L^{-1}_{\alpha\alpha} = r\langle\alpha|\frac{1}{\kappa I - A}|\alpha\rangle = r\int_{R-\{\kappa\}}\frac{d\mu(x)}{\kappa - x} = r\sum_{i,i\neq 0}^{d-1}\frac{A_i}{\kappa - x_i} = r\lim_{y\to\kappa}(G_\mu(y) - \frac{A_0}{y-\kappa}) \tag{3-34}$$

and

$$\begin{aligned} L^{-1}_{\beta\alpha} &= r\langle\beta|\frac{1}{\kappa I - A}|\alpha\rangle = \frac{r}{\sqrt{\kappa_l}}\langle\phi_l|\frac{1}{\kappa I - A}|\alpha\rangle = \frac{r}{\sqrt{\kappa_l}}\langle\alpha|\frac{P_l(A)}{\kappa I - A}|\alpha\rangle \\ &= \frac{r}{\sqrt{\kappa_l}}\int_{R-\{\kappa\}}\frac{d\mu(x)}{\kappa - x}P_l(x) = \frac{r}{\sqrt{\kappa_l}}\sum_{i,i\neq 0}\frac{A_iP_l(x_i)}{\kappa - x_i}, \end{aligned} \tag{3-35}$$

where, we have considered $x_0 = \kappa$ (κ is the eigenvalue corresponding to the idempotent E_0). Then, by using (3-33), the two-point resistance $R_{\alpha\beta^{(l)}}$ in the network is given by

$$R_{\alpha\beta^{(l)}} = \frac{2r}{\sqrt{\kappa_l}}(\sqrt{\kappa_l}\lim_{y\to\kappa}(G_\mu(y) - \frac{A_0}{y-\kappa}) - \sum_{i,i\neq 0}\frac{A_iP_l(x_i)}{\kappa - x_i}). \tag{3-36}$$

For evaluating the term $\sum_{i,i\neq 0}\frac{A_iP_l(x_i)}{\kappa - x_i}$ in (3-36), we need to calculate

$$I_m := \sum_{i,i\neq 0}\frac{A_ix_i^m}{\kappa - x_i}, \quad \text{for} \quad m = 0, 1, ..., l. \tag{3-37}$$

To do so, we write the term (3-37) as

$$\begin{aligned} I_m &= \sum_{i,i\neq 0}\frac{A_ix_i^m}{\kappa - x_i} = \sum_{i,i\neq 0}\frac{A_i((x_i-\kappa)^m - \sum_{l=1}^m(-1)^lC_l^m\kappa^lx_i^{m-l})}{\kappa - x_i} \\ &= -\sum_{i,i\neq 0}A_i(x_i-\kappa)^{m-1} - \sum_{l=1}^m(-1)^lC_l^m\kappa^l\sum_{i,i\neq 0}\frac{A_ix_i^{m-l}}{\kappa - x_i}, \end{aligned} \tag{3-38}$$

that is, we have

$$I_m = -\sum_{l=0}^{m-1}(-1)^lC_l^{m-1}\kappa^l\sum_{i,i\neq 0}A_ix_i^{m-l-1} - \sum_{l=1}^m(-1)^lC_l^m\kappa^lI_{m-l}. \tag{3-39}$$

Therefore, I_m can be calculated recursively, if we able to calculate the term $\sum_{i,i\neq 0} A_i x_i^{m-l-1}$ for $l = 0, 1, ..., m-1$ appearing in (3-38). For example, for $m = 1$, we obtain

$$I_1 = \sum_{i,i\neq 0} \frac{A_i x_i}{\kappa - x_i} = -\sum_{i,i\neq 0} A_i + \kappa I_0 = -1 + A_0 + \kappa \sum_{i,i\neq 0} \frac{A_i}{\kappa - x_i}. \tag{3-40}$$

In order to evaluation of the sum $\sum_{i,i\neq 0} A_i x_i^k$, we rescale the roots x_i as ξx_i, where ξ is some nonzero constant. Then, we will have

$$\frac{1}{\xi} G_\mu(x/\xi) = \sum_i \frac{A_i}{x - \xi x_i} + \frac{A_0}{x - \xi x_0}. \tag{3-41}$$

Now, we take the m-th derivative of (3-41) to obtain

$$\frac{\partial^m}{\partial \xi^m}(\frac{1}{\xi} G_\mu(x/\xi)) = m!(\sum_{i,i\neq 0} \frac{A_i x_i^m}{(x - \xi x_i)^{m+1}} + \frac{A_0 x_0^m}{(x - \xi x_0)^{m+1}}), \tag{3-42}$$

where, at the limit of the large x, one can obtain the following simple form

$$\lim_{x\to\infty} \frac{\partial^m}{\partial \xi^m}(\frac{1}{\xi} G_\mu(x/\xi)) = m!(\frac{\sum_{i,i\neq 0} A_i x_i^m + A_0 x_0^m}{x^{m+1}}) \tag{3-43}$$

Therefore, we obtain

$$\sum_{i,i\neq 0} A_i x_i^m = \frac{1}{m!} \lim_{x\to\infty} [x^{m+1} \frac{\partial^m}{\partial \xi^m}(\frac{1}{\xi} G_\mu(x/\xi))] - A_0 x_0^m. \tag{3-44}$$

3.1 Two-point resistances up to the third stratum

In this subsection we give analytical formulas for two-point resistances $R_{\alpha\beta^{(i)}}, i = 1, 2, 3$, in terms of intersection numbers.

It should be noticed that, for two arbitrary nodes α and β such that $\beta \in R_1(\alpha)$, we have $P_1(x) = \frac{x}{\sqrt{\kappa}}$. Therefore, by using (3-35) and (3-40) we obtain

$$L^{-1}_{\beta\alpha} = \frac{r}{\kappa} \sum_{i,i\neq 0} \frac{A_i x_i}{\kappa - x_i} = \frac{-r}{\kappa} \sum_{i,i\neq 0} A_i + r \sum_{i,i\neq 0} \frac{A_i}{\kappa - x_i}. \tag{3-45}$$

Therefore, by using (3-33), we obtain the following simple result for all $\beta \in R_1(\alpha)$

$$R_{\alpha\beta^{(1)}} = \frac{2r}{\kappa} \sum_{i,i\neq 0} A_i = \frac{2r}{\kappa}(1 - A_0) = \frac{2r}{\kappa}(1 - \frac{1}{v}), \tag{3-46}$$

where, v is the number of vertices of the graph, and in the last equality we have used the fact that for regular graphs, we have

$$A_0 = \frac{1}{v}. \tag{3-47}$$

In the following, we give analytical formulas for calculating two-point resistances $R_{\alpha\beta^{(2)}}$ and $R_{\alpha\beta^{(3)}}$, where $R_{\alpha\beta^{(2)}}(R_{\alpha\beta^{(3)}})$ denotes the mutual resistances between α and all $\beta \in R_2(\alpha)$ (all $\beta \in R_3(\alpha)$).

By using (2-22) and that $P_k = \frac{Q_k}{\sqrt{\omega_1 \ldots \omega_k}}$, we have $P_2(x) = \frac{1}{\sqrt{\omega_1\omega_2}}(x^2 - \alpha_1 x - \omega_1)$. Then, from (3-35) after some simplifications we obtain for $\beta \in R_2(\alpha)$

$$L^{-1}_{\alpha\beta^{(2)}} = \frac{r}{\sqrt{\omega_1\omega_2\kappa_2}}(-\sum_{i,i\neq 0} A_i x_i + (\alpha_1 - \kappa)\sum_{i,i\neq 0} A_i + \kappa(\kappa - \alpha_1 - 1)\sum_{i,i\neq 0}\frac{A_i}{\kappa - x_i}). \tag{3-48}$$

By substituting $\alpha_1 = \kappa - b_1 - c_1$ in $\kappa(\kappa - \alpha_1 - 1)$, we obtain

$$\kappa(\kappa - \alpha_1 - 1) = \kappa(b_1 + c_1 - 1) = \kappa b_1. \tag{3-49}$$

Then, the coefficient of the term $\sum_{i,i\neq 0}\frac{A_i}{\kappa - x_i}$ in (3-48) is

$$\frac{r\kappa b_1}{\sqrt{\omega_1\omega_2\kappa_2}} = \frac{r\kappa b_1}{\sqrt{\kappa b_1 c_2 \kappa_2}} = r\sqrt{\frac{\kappa b_1}{c_2\kappa_2}} = r. \tag{3-50}$$

Therefore, (3-48) can be written as

$$L^{-1}_{\alpha\beta^{(2)}} = \frac{r}{\sqrt{\omega_1\omega_2\kappa_2}}(-\sum_{i,i\neq 0} A_i x_i + (\alpha_1 - \kappa)\sum_{i,i\neq 0} A_i) + r\sum_{i,i\neq 0}\frac{A_i}{\kappa - x_i}, \tag{3-51}$$

where, the sum $\sum_{i,i\neq 0} A_i x_i$ can be calculated by using (3-44). It can be easily shown that

$$\lim_{x\to\infty}[x^2\frac{\partial}{\partial\xi}(\frac{1}{\xi}G_\mu(x/\xi))] = a_{d-2} - b_{d-1}, \tag{3-52}$$

where, a_{d-2} and b_{d-1} are the coefficients of x^{d-2} and x^{d-1} in $Q^{(1)}_{d-1}$ and Q_d, respectively. From the recursion relations (2-22) and (2-24), one can see that $a_{d-2} = b_{d-1} = -(\alpha_1 + \ldots + \alpha_d)$. Therefore, from (3-44) and (3-52) we obtain

$$\sum_{i,i\neq 0} A_i x_i = -A_0\kappa = -\frac{\kappa}{v}. \tag{3-53}$$

Then, by using (3-32) and (3-51), one can write $R_{\alpha\beta^{(2)}}$ as follows

$$R_{\alpha\beta^{(2)}} = \frac{2r}{\sqrt{\omega_1\omega_2\kappa_2}}\{(\kappa-\alpha_1) - \frac{2\kappa-\alpha_1}{v}\}, \tag{3-54}$$

where, by using (2-11) and (2-13), we obtain the following main result in terms of the intersection numbers of the graph

$$R_{\alpha\beta^{(2)}} = \frac{2r}{b_0 b_1}\{b_1 + 1 - \frac{b_0+b_1+1}{v}\}. \tag{3-55}$$

Now, consider $\beta \in R_3(\alpha)$. Then, by using (2-22) and $P_k = \frac{Q_k}{\beta_1 \ldots \beta_k}$ we obtain $P_3(x) = \frac{1}{\sqrt{\omega_1\omega_2\omega_3}}(x^3 - (\alpha_1+\alpha_2)x^2 - (\omega_1+\omega_2-\alpha_1\alpha_2)x + \alpha_2\omega_1)$. As above, after some calculations, we obtain for $\beta \in R_3(\alpha)$

$$L^{-1}_{\alpha\beta^{(3)}} = \frac{r}{\sqrt{\omega_1\omega_2\omega_3\kappa_3}}\{\frac{\kappa^2}{v} - 2(a_{d-3} - b_{d-2} + b^2_{d-1} - b_{d-1}a_{d-2}) - (\alpha_1+\alpha_2-\kappa)\frac{\kappa}{v} -$$
$$(\kappa^2 - \kappa(\alpha_1+\alpha_2) - \omega_1 - \omega_2 + \alpha_1\alpha_2)(\frac{v-1}{v}) + (\kappa^3 - \kappa^2(\alpha_1+\alpha_2) - \kappa(\omega_1+\omega_2-\alpha_1\alpha_2) + \alpha_2\omega_1)\sum_{i,i\neq 0}\frac{A_i}{\kappa - x_i}\}. \tag{3-56}$$

Again, by substituting α_1, α_2, ω_1 and ω_2 from (2-13), we have

$$\frac{1}{\sqrt{\omega_1\omega_2\omega_3\kappa_3}}(\kappa^3 - \kappa^2(\alpha_1+\alpha_2) - \kappa(\omega_1+\omega_2-\alpha_1\alpha_2) + \alpha_2\omega_1) = \frac{\kappa b_1 b_2}{\sqrt{\kappa b_1 c_2 b_2 c_3 \kappa_3}} = 1. \tag{3-57}$$

Therefore, (3-56) can be written as follows

$$L^{-1}_{\alpha\beta^{(3)}} = \frac{r}{\sqrt{\omega_1\omega_2\omega_3\kappa_3}}\{\frac{\kappa^2}{v} - 2(a_{d-3} - b_{d-2} + b^2_{d-1} - b_{d-1}a_{d-2}) - (\alpha_1+\alpha_2-\kappa)\frac{\kappa}{v} -$$
$$(\kappa^2 - \kappa(\alpha_1+\alpha_2) - \omega_1 - \omega_2 + \alpha_1\alpha_2)(\frac{v-1}{v})\} + r\sum_{i,i\neq 0}\frac{A_i}{\kappa - x_i}. \tag{3-58}$$

In (3-56), we have used the following equality

$$\lim_{x^2\to\infty}[x^3\frac{\partial^2}{\partial\xi^2}(\frac{1}{\xi}G_\mu(x/\xi))] = 2(a_{d-3} - b_{d-2} + b^2_{d-1} - b_{d-1}a_{d-2}), \tag{3-59}$$

where, a_{d-3} and b_{d-2} are the coefficients of x^{d-3} and x^{d-2} in $Q^{(1)}_{d-1}$ and Q_d, respectively. From the recursion relations (2-22) and (2-24), one can see that $a_{d-3} = \prod_{i<j=1}^{d}\alpha_i\alpha_j - (\omega_1 + \ldots + \omega_{d-1})$ and $b_{d-2} = a_{d-3} - \omega_d$. Therefore, we have $a_{d-3} - b_{d-2} + b^2_{d-1} - b_{d-1}a_{d-2} = \omega_d$.

Then, by using (3-32), $R_{\alpha\beta^{(3)}}$ is given by

$$R_{\alpha\beta^{(3)}}=\frac{2r}{\sqrt{\omega_1\omega_2\omega_3\kappa_3}}\{\omega_d+(\alpha_1+\alpha_2-2\kappa)\frac{\kappa}{v}+(\kappa^2-\kappa(\alpha_1+\alpha_2)-\omega_1-\omega_2+\alpha_1\alpha_2)(\frac{v-1}{v})\}. \quad (3\text{-}60)$$

In terms of the intersection numbers of the graph, we obtain the following main result

$$R_{\alpha\beta^{(3)}}=\frac{2r}{b_0b_1b_2}\{b_{d-1}c_d+b_2-b_0+c_2+b_1b_2-\frac{(b_0+1)(b_2+c_2)+b_1(b_0+b_2)}{v}\}. \quad (3\text{-}61)$$

3.2 Two-point resistances in infinite regular networks

As the results (3-33) and (3-44) show, the two-point resistances on a network depend only on the Stieltjes function $G_\mu(x)$ corresponding to the network. Clearly, each Stieltjes function corresponding to an infinite network has a unique representation as an infinite continued fraction as follows

$$G_\mu(x)=\int_R\frac{\mu(dy)}{x-y}=\frac{1}{z-\alpha_0-\frac{\beta_1^2}{z-\alpha_1-\frac{\beta_2^2}{z-\alpha_2-\frac{\beta_3^2}{z-\alpha_3-\cdots}}}}, \quad (3\text{-}62)$$

where, the sequence $\alpha_0,\alpha_1,...;\beta_1,\beta_2,...$ never stops. It is well known that all infinite continued fraction expansions as in (3-62) can be approximated with finite ones. Therefore, the two-point resistances on an infinite-size resistor network can be approximated with those of the corresponding finite-size networks.

In the following section, we give some examples of finite resistor networks such that at the limit of the large size of the networks, we obtain some infinite regular networks. Then, we approximate the infinite networks with finite ones as illustrated above.

4 Examples

In this section we calculate two-point resistances $R_{\alpha\beta^{(i)}}$, for $i=1,2,3$ by using (3-46), (3-55) and (3-61), in some important distance-regular networks with diameters $d=1$, $d=2$ and $d>2$, respectively..

4.1 Complete network K_n

The complete network K_n is the simplest example of distance-regular networks. This network has n vertices with $n(n-1)/2$ edges, the degree of each vertex is $\kappa = n-1$ also the network has diameter $d = 1$. The intersection array of the network is $\{c_0; b_1\} = \{n-1; 1\}$. Clearly, this graph has only two strata $\Gamma_0(\alpha) = \alpha$ and $R_1(\alpha) = \{\beta : \beta \neq \alpha\}$. Then, the only two-point resistance is $R_{\alpha\beta^{(1)}}$ which is given by using (3-46) as follows

$$R_{\alpha\beta^{(1)}} = \frac{2r}{v-1}(1-\frac{1}{v}) = \frac{2r}{v} \quad \text{for all} \ \ \beta \in \Gamma_1(\alpha). \tag{4-63}$$

4.2 Strongly regular networks

One of the most important distance regular networks are those with diameter $d = 2$, called strongly regular networks. A network with v vertices is strongly regular with parameters v, κ, λ, μ whenever it is not complete or edgeless and

(i) each vertex is adjacent to κ vertices,

(ii) for each pair of adjacent vertices there are λ vertices adjacent to both,

(iii) for each pair of non-adjacent vertices there are μ vertices adjacent to both.

For a strongly regular network, the intersection array is given by

$$\{c_0, c_1; b_1, b_2\} = \{\kappa, \kappa-\lambda-1; 1, \mu\}. \tag{4-64}$$

One can notice that, if we consider networks with diameter two and maximum degree κ and $\alpha \in V$, then α has at most κ neighbors, and at most $\kappa(\kappa-1)$ vertices lie at distance two from α. Therefore

$$v \leq 1 + \kappa + \kappa^2 - \kappa = \kappa^2 + 1, \quad \text{or} \quad \kappa \geq \sqrt{v-1}, \tag{4-65}$$

where, in the following by using the inequality (4-65), we will obtain upper bounds for two-point resistances in strongly regular networks. To do so, first we calculate two-point resistances for these networks.

By using (3-46), (3-55) and (4-64), we obtain

$$R_{\alpha\beta^{(1)}} = \frac{2r}{\kappa}(\frac{v-1}{v}), \quad \text{and} \tag{4-66}$$

$$R_{\alpha\beta^{(2)}} = \frac{2r}{\kappa(\kappa-\lambda-1)}(\kappa-\lambda-\frac{2\kappa-\lambda}{v}), \tag{4-67}$$

respectively. Then, from (4-65) and (4-66), we obtain the following upper bound for $R_{\alpha\beta^{(1)}}$

$$R_{\alpha\beta^{(1)}} \leq \frac{2r\sqrt{v-1}}{v}. \tag{4-68}$$

Now, we consider the following two well-known strongly regular networks.

A. Petersen network

Petersen network [13] is a strongly regular network with parameters $(v, \kappa, \lambda, \eta) = (10, 3, 0, 1)$ and the intersection array $\{c_0, c_1; b_1, b_2\} = \{3, 2; 1, 1\}$. Therefore, by using (4-66) and (4-67), we obtain

$$R_{\alpha\beta^{(1)}} = \frac{3r}{5} \quad \text{and} \quad R_{\alpha\beta^{(2)}} = \frac{4r}{5}. \tag{4-69}$$

From (4-69), it is seen that $R_{\alpha\beta^{(1)}}$ in Petersen graph saturates the upper bound (4-68).

B. Normal subgroup scheme

Definition 2.3 The partition $P = \{P_0, P_1, ..., P_d\}$ of a finite group G is called a blueprint [13] if

(i) $P_0 = \{e\}$

(ii) for i=1,2,...,d if $g \in P_i$ then $g^{-1} \in P_i$

(iii) the set of relations $R_i = \{(\alpha, \beta) \in G \otimes G | \alpha^{-1}\beta \in P_i\}$ on G form an association scheme.

The set of real conjugacy classes given in Appendix A of Ref. [1] is an example of blueprint on G. Also, one can show that in the regular representation, the class sums $\bar{P}_i$ for $i = 0, 1, ..., d$ defined as

$$\bar{P}_i = \sum_{\gamma \in P_i} \gamma \in CG, \quad i = 0, 1, ..., d, \tag{4-70}$$

are the adjacency matrices of a blueprint scheme.

In [1], it has been shown that, if H be a normal subgroup of G, the following blueprint classes

$$P_0 = \{e\}, \quad P_1 = G - \{H\}, \quad P_2 = H - \{e\}, \tag{4-71}$$

define a strongly regular network with parameters $(v, \kappa, \lambda, \eta) = (g, g-h, g-2h, g-h)$ and the following intersection array

$$\{c_0, c_1; b_1, b_2\} = \{g-h, h-1; 1, g-h\}, \tag{4-72}$$

where, $g := |G|$ and $h := |H|$. It is interesting to note that in normal subgroup scheme, the intersections numbers and other parameters depend only on the cardinalities of the group and its normal subgroup. By using (2-13), the QD parameters are given by $\{\alpha_1, \alpha_2; \omega_1, \omega_2\} = \{g-2h, 0; g-h, (g-h)(h-1)\}$. Also, from (4-71), it is seen that $|\Gamma_2(\alpha)| = h-1$. Then, by using (4-66) and (4-67), we obtain

$$R_{\alpha\beta^{(1)}} = \frac{2r(g-1)}{g(g-h)}, \quad \text{and} \quad R_{\alpha\beta^{(2)}} = \frac{2r}{(g-h)}. \tag{4-73}$$

One should notice that, the maximum degree κ for normal subgroup scheme is $\kappa_{max} = g-2$ ($h = 2$), which can be appear in networks with even cardinality such as dihedral group. Therefore, for normal subgroup scheme (strongly regular networks with parameters $(g, g-h, g-2h, g-h)$), we have

$$\kappa \leq g-2, \tag{4-74}$$

and therefore, by using (4-65), (4-78) and (4-75) ($\kappa = g-h$), we obtain upper and lower bounds for $R_{\alpha\beta^{(1)}}$ and $R_{\alpha\beta^{(2)}}$ as follows

$$\frac{2r(g-1)}{g(g-2)} \leq R_{\alpha\beta^{(1)}} \leq \frac{2r\sqrt{g-1}}{g} \quad , \qquad \frac{2r}{g-2} \leq R_{\alpha\beta^{(2)}} \leq \frac{2r}{\sqrt{g-1}}. \tag{4-75}$$

As an example, we consider the dihedral group $G = D_{2m}$, where its normal subgroup is $H = Z_m$. Therefore, the blueprint classes are given by

$$P_0 = \{e\}, \quad P_1 = \{b, ab, a^2b, ..., a^{m-1}b\}, \quad P_2 = \{a, a^2, ..., a^{(m-1}\}, \tag{4-76}$$

which form a strongly regular network with parameters $(2m, m, 0, m)$ and the following intersection array

$$\{c_0, c_1; b_1, b_2\} = \{m, m-1; 1, m\}. \tag{4-77}$$

By using (4-78), we obtain

$$R_{\alpha\beta^{(1)}} = \frac{r(2m-1)}{m^2}, \quad \text{and} \quad R_{\alpha\beta^{(2)}} = \frac{2r}{m}. \tag{4-78}$$

4.3 Cycle network C_v

A cycle network or cycle is a network that consists of some number of vertices connected in a closed chain. The cycle network with v vertices is denoted by C_v with $\kappa = 2$. For odd $v = 2m+1$, the intersection array is given by

$$\{c_0, ..., c_m; b_1, ..., b_m\} = \{2, 1, ..., 1; 1, ..., 1, 1\}, \tag{4-79}$$

where, for even $v = 2m$, we have

$$\{c_0, ..., c_m; b_1, ..., b_m\} = \{2, 1, ..., 1, 2; 1, ..., 1, 2\} \tag{4-80}$$

and the network consists of $m+1$ strata. We consider $v = 2m$ (the case $v = 2m+1$ can be considered similarly). Then, by using the recursion relations (2-22) and (2-24) one can obtain the following closed form for the Stieltjes function

$$G_\mu(x) = \frac{1}{m}\frac{T'_m(x/2)}{T_m(x/2)}, \tag{4-81}$$

where, $T_k(x)$ are Tchebyshev polynomials of the first kind.

By using (3-46), (3-55) and (3-61) we obtain

$$R_{\alpha\beta^{(1)}} = r(\frac{2m-1}{2m}), \quad R_{\alpha\beta^{(2)}} = 2r(\frac{m-1}{m}), \quad \text{and} \quad R_{\alpha\beta^{(3)}} = 3r(\frac{2m-3}{2m}), \tag{4-82}$$

respectively. From (4-82), one can easily deduce that

$$R_{\alpha\beta^{(k)}} = kr(\frac{2m-k}{2m}) \qquad k = 1, 2, ..., m. \tag{4-83}$$

At the limit of the large m, the cycle network tend to the infinite line network and the Stieltjes function (4-81) reads as

$$G_\mu(x) = \frac{1}{\sqrt{x^2-4}}. \tag{4-84}$$

Then, for A_0 we have

$$A_0 = \lim_{x\to 2}((x-2)G_\mu(x)) = 0. \tag{4-85}$$

Therefore, by using (3-46), (3-55) and (3-61) we obtain

$$R_{\alpha\beta^{(1)}} = r, \quad R_{\alpha\beta^{(2)}} = 2r, \quad R_{\alpha\beta^{(3)}} = 3r. \tag{4-86}$$

In fact, it can be easily shown that

$$R_{\alpha\beta^{(k)}} = kr, \quad k = 1, 2, ... \tag{4-87}$$

where, this result could be obtained from (4-83), for large m.

As (4-83) indicates, for m larger than ~ 70 the difference $|R_{\alpha\beta^{(k)}} - kr|$ is equal to ~ 0.01, where for m larger than ~ 100 we have $|R_{\alpha\beta^{(k)}} - kr| \sim 0$. Therefore, the finite resistor network C_{2m}, with $m \sim 100$ is a good approximation for the infinite line resistor network.

4.4 d-cube

The d-cube, i.e. the hypercube of dimension d, also called Hamming cubes, is a network with 2^d nodes, each of which can be labeled by an d-bit binary string. Two nodes on the hypercube described by bitstrings $\vec{x}$ and $\vec{y}$ are are connected by an edge if $|\vec{x} - \vec{y}| = 1$, where $|\vec{x}|$ is the Hamming weight of $\vec{x}$. In other words, if $\vec{x}$ and $\vec{y}$ differ by only a single bit flip, then the two corresponding nodes on the graph are connected. Thus, each of the 2^d nodes on the d-cube has degree d. For the d-cube we have $d+1$ strata with

$$\kappa_i = \frac{d!}{i!(d-i)!} \ , \quad 0 \le i \le d-1. \tag{4-88}$$

The intersection numbers are given by

$$b_i = d - i, \quad 0 \le i \le d-1; \quad c_i = i, \quad 1 \le i \le d. \tag{4-89}$$

Then, by using (3-46), (3-55) and (3-61), we obtain

$$R_{\alpha\beta^{(1)}} = \frac{2^d - 1}{d2^{d-1}} r, \quad R_{\alpha\beta^{(2)}} = \frac{2^{d-1} - 1}{(d-1)2^{d-2}} r, \quad \text{and}$$

$$R_{\alpha\beta^{(3)}} = \frac{r}{d(d-1)(d-2)} \{ \frac{2^d(d^2 - 2d + 2) - 3d(d-1) - 2}{2^{d-1}} \}. \tag{4-90}$$

From (4-90) one can see that, at the limit of the large dimension d, the two-point resistances $R_{\alpha\beta^{(i)}}$, $i = 1, 2, 3$ tend to zero. Since $R_{\alpha\beta^{(1)}}$ tend to zero for d larger than ~ 200. Therefore, the finite d-cube with d larger than ~ 200 is a good approximation for the infinite hypercube resistor network.

4.5 Johnson network

Let $n \ge 2$ and $d \le n/2$. The Johnson network $J(n, d)$ has all d-element subsets of $\{1, 2, ..., n\}$ such that two d-element subsets are adjacent if their intersection has size $d-1$. Two d-element subsets are then at distance i if and only if they have exactly $d - i$ elements in common. The Johnson network $J(n, d)$ has $v = \frac{n!}{d!(n-d)!}$ vertices, diameter d and the valency $\kappa = d(n - d)$. Its intersection array is given by

$$b_i = (d - i)(n - d - i), \quad 0 \le i \le d - 1; \quad c_i = i^2, \quad 1 \le i \le d, \tag{4-91}$$

Then, by using (3-46), (3-55) and (3-61), we obtain

$$R_{\alpha\beta^{(1)}} = \frac{2(n! - d!(n-d)!)}{d(n-d)n!} r,$$

$$R_{\alpha\beta^{(2)}} = \frac{2r}{d(d-1)(n-d)(n-d-1)} \{ d(n-d) - (n-2) + \frac{d!(n-d)!(n-2-2d(n-d))}{n!} \} \quad \textit{and}$$

$$R_{\alpha\beta^{(3)}} = \frac{2r}{d(d-1)(d-2)(n-d)(n-d-1)(n-d-2)} \{ d^2(n - 2d + 1) +$$

$$(3n-2d(n-d)-10)\frac{d(n-d)d!(n-d)!}{n!}+[d^2(n-d)^2-d(n-d)(3n-9)-4(d-1)(n-d-1)+$$
$$2(n-2)(n-4)](1-\frac{d!(n-d)!}{n!})\}. \qquad (4\text{-}92)$$

It could be noticed that for a give d, the result (4-92) show that, at the limit of the large dimension n, the two-point resistances $R_{\alpha\beta^{(i)}}$, $i = 1, 2, 3$ tend to zero. Since $R_{\alpha\beta^{(1)}}$ tend to zero for n larger than ~ 200. Therefore, the finite Johnson network $J(n, d)$ with n larger than ~ 200 is a good approximation for the infinite Johnson resistor network.

5 Two-point resistances in more general networks

Although, we discussed through the paper only the case of distance-regular networks, but also the method can be used for any arbitrary regular network. For calculating two-point resistances, we need only to know the Stieltjes function $G_\mu(x)$. For two arbitrary nodes α and β of the network, we choose one of the nodes, say α, as reference vertex. Then, the Stieltjes function $G_\mu(x)$ can be calculated by using the recursion relations (2-22) and (2-24), where, as it has be shown in [20], the coefficients α_i and β_i, for $i = 1, ..., d$ in the recursion relations are obtained by applying the Lanczos algorithm to the adjacency matrix of the network and the reference vertex $|\alpha\rangle$. In fact, the adjacency matrix of the network takes a tridiagonal form in the orthonormal basis $\{|\phi_i\rangle, i = 0, 1, ..., d\}$ produced by Lanczos algorithm and so we obtain again three term recursion relations as (2-22). But, in general, the basis produced by Lanczos algorithm do not define a stratification basis, in the sense that, a vertex ket $|\beta\rangle$ of the network may be appear in more than one of the base vectors $|\phi_i\rangle$. In these cases, if d is equal to v (the number of vertices of the network), one can write each vertex ket $|\beta\rangle$ uniquely as a superposition of the base vectors $|\phi_i\rangle$ and calculate two-point resistance $R_{\alpha\beta}$ by calculating the entries $\langle\phi_i|L^{-1}|\alpha\rangle$ for all $i = 0, 1, ..., d$ as illustrated through the paper. In the most cases, d is less than v. In these cases, we need to obtain some additional orthonormal base vectors $\{|\psi_i\rangle, i = 1, .., v-d-1\}$ such that the new bases are orthogonal to the subspace spanned

by $\{|\phi_i\rangle, i = 0, 1, ..., d\}$. One can obtain some such additional base vectors, by choosing a normalized vector orthogonal to the subspace spanned by $\{|\phi_i\rangle, i = 0, 1, ..., d\}$ as a new reference state and applying the Lanczos algorithm to the adjacency matrix of the network and the new reference state. If the number of the new orthonormal base vectors still be less than $v-d-1$, we choose another normalized reference state orthogonal to the subspace spanned by all previous orthonormal bases and apply the Lanczos algorithm to the adjacency matrix and the new chosen reference state. By repeating this process until to obtain v orthonormal basis, one can solve a system of v equations with v unknowns to write each vertex ket β as superposition of the v orthonormal bases.

6 Conclusion

The resistance between two arbitrary nodes in a distance-regular resistor network was obtained in terms of the Stieltjes transform of the spectral measure or Stieltjes function associated with the network and its derivatives. It was shown that the resistances between a node α and all nodes β belonging to the same stratum with respect to the α are the same. Also, explicit analytical formulaes for two-point resistances $R_{\alpha\beta}$ for β belonging to the first, second and third stratum with respect to the α were driven in terms of the size of the network and the corresponding intersection numbers. In particular, the two-point resistances in a strongly regular network with parameters $(v, \kappa, \lambda, \mu)$ were given in terms of these parameters. Moreover, the lower and upper bounds for two-point resistances in strongly regular networks was discussed. It was discussed that, the introduced method can be used not only for distance-regular networks, but also for any arbitrary regular network by employing the Lanczos algorithm iteratively.

Appendix A

In this appendix, we give the two-point resistances $R_{\alpha\beta(i)}$, $i = 1,2,3$ for some important distance-regular networks with $v \leq 70$.

The network	v	Intersection array	Ref.	$R_{\alpha\beta(1)}$	$R_{\alpha\beta(2)}$	$R_{\alpha\beta(3)}$
Icosahedron	12	$\{5,2,1;1,2,5\}$	[21]	$\frac{11r}{30}$	$\frac{7r}{15}$	$\frac{r}{2}$
L(Petersen)	15	$\{4,2,1;1,1,4\}$	[21]	$\frac{7r}{15}$	$\frac{19r}{30}$	$\frac{2r}{3}$
Pappus, 3-cover $K_{3,3}$	18	$\{3,2,2,1;1,1,2,3\}$	[21]	$\frac{17r}{27}$	$\frac{8r}{9}$	$\frac{26r}{27}$
Desargues	20	$\{3,2,2,1,1;1,1,2,2,3\}$	[21]	$\frac{19r}{30}$	$\frac{8r}{9}$	$\frac{59r}{60}$
Dodecahedron	20	$\{3,2,1,1,1;1,1,1,2,3\}$	[21]	$\frac{19r}{30}$	$\frac{9r}{10}$	$\frac{16r}{15}$
$GH(2,1)$	21	$\{4,2,2;1,1,2\}$	[21]	$\frac{10r}{21}$	$\frac{2r}{3}$	$\frac{5r}{7}$
Klein	24	$\{7,4,1;1,2,7\}$	[21]	$\frac{23r}{84}$	$\frac{9r}{28}$	$\frac{r}{3}$
$GQ(2,4)\backslash$ spread	27	$\{8,6,1;1,3,8\}$	[21]	$\frac{13r}{54}$	$\frac{29r}{108}$	$\frac{5r}{18}$
$H(3,3)$	27	$\{6,4,2;1,2,3\}$	[21]	$\frac{26r}{81}$	$\frac{31r}{81}$	$\frac{11r}{27}$
coxeter	28	$\{3,2,2,1;1,1,1,2\}$	[21]	$\frac{9r}{14}$	$\frac{13r}{14}$	$\frac{73r}{84}$
Taylor($P(13)$)	28	$\{13,6,1;1,6,13\}$	[22]	$\frac{27r}{182}$	$\frac{44r}{273}$	$\frac{91r}{546}$
Tutte's 8-cage	30	$\{3,2,2,2;1,1,1,3\}$	[21]	$\frac{29r}{45}$	$\frac{14r}{15}$	$\frac{139r}{90}$
Taylor(GQ(2,2))	32	$\{15,8,1;1,8,15\}$	[22]	$\frac{31r}{240}$	$\frac{11r}{80}$	$\frac{17r}{120}$
Taylor($T(6)$)	32	$\{15,6,1;1,6,15\}$	[22]	$\frac{31r}{240}$	$\frac{101r}{720}$	$\frac{13r}{90}$
$IG(AG(2,4)\setminus pc)$	32	$\{4,3,3,1;1,1,3,4\}$	[23]	$\frac{31r}{64}$	$\frac{5r}{8}$	$\frac{125r}{192}$
Wells	32	$\{5,4,1,1;1,1,4,5\}$	[23]	$\frac{31r}{80}$	$\frac{15r}{32}$	$\frac{39r}{80}$
Hadamard graph	32	$\{8,7,4,1;1,4,7,8\}$	[24]	$\frac{31r}{128}$	$\frac{15r}{56}$	$\frac{249r}{896}$
Odd(4)	35	$\{4,2,1;1,1,4\}$	[25],[26]	$\frac{17r}{35}$	$\frac{22r}{35}$	$\frac{242r}{315}$
Sylvester	36	$\{5,4,4;1,1,4\}$	[25]	$\frac{7r}{18}$	$\frac{17r}{36}$	$\frac{181r}{240}$
Taylor($P(17)$)	36	$\{17,8,1;1,8,17\}$	[22]	$\frac{35r}{306}$	$\frac{149r}{1224}$	$\frac{279r}{272}$
3-cover $K_{6,6}$	36	$\{6,5,4,1;1,2,5,6\}$	[22]	$\frac{35r}{108}$	$\frac{17r}{45}$	$\frac{211r}{540}$

The network	v	Intersection array	Ref.	$R_{\alpha\beta^{(1)}}$	$R_{\alpha\beta^{(2)}}$	$R_{\alpha\beta^{(3)}}$
$SRG\backslash$ spread	40	$\{9,6,1;1,2,9\}$	[27]	$\frac{13r}{60}$	$\frac{11r}{45}$	$\frac{r}{4}$
$Ho-Si_2(x)$	42	$\{16,5,1;1,1,6\}$	[25]	$\frac{41r}{126}$	$\frac{8r}{21}$	$\frac{34r}{105}$
Mathon (Cycl$(13,3)$)	42	$\{13,8,1;1,4,13\}$	[28]	$\frac{41r}{273}$	$\frac{89r}{546}$	$\frac{91r}{13}$
$GO(2,1)$	45	$\{4,2,2,2;1,1,1,2\}$	[23]	$\frac{22r}{45}$	$\frac{32r}{45}$	$\frac{4r}{5}$
3-cover $GQ(2,2)$	45	$\{6,4,2,1;1,1,4,6\}$	[23]	$\frac{44r}{135}$	$\frac{107r}{270}$	$\frac{221r}{540}$
Hadamard graph	48	$\{12,11,6,1;1,6,11,12\}$	[22]	$\frac{47r}{288}$	$\frac{23r}{132}$	$\frac{565r}{3168}$
$IG(AG(2,5)\setminus pc)$	50	$\{5,4,4,1;1,1,4,5\}$	[23]	$\frac{49r}{125}$	$\frac{12r}{25}$	$\frac{123r}{250}$
Mathon (Cycl$(16,3)$)	51	$\{16,10,1;1,5,16\}$	[28]	$\frac{25r}{204}$	$\frac{89r}{680}$	$\frac{68r}{510}$
$GH(3,1)$	52	$\{6,3,3;1,1,2\}$	[29]	$\frac{51r}{156}$	$\frac{11r}{26}$	$\frac{23r}{52}$
Taylor(SRG$(25,12)$)	52	$\{25,12,1;1,12,25\}$	[22]	$\frac{51r}{650}$	$\frac{319r}{3900}$	$\frac{13r}{156}$
3-cover $K_{9,9}$	54	$\{9,8,6,1;1,3,8,9\}$	[22]	$\frac{53r}{243}$	$\frac{13r}{54}$	$\frac{239r}{972}$
Gosset,Tayl(Schl$\ddot{a}$fli)	56	$\{27,10,1;1,10,27\}$	[22]	$\frac{55r}{756}$	$\frac{289r}{3780}$	$\frac{49r}{630}$
Taylor(Co-Schl$\ddot{a}$fli)	56	$\{27,16,1;1,16,27\}$	[22]	$\frac{55r}{756}$	$\frac{227r}{3024}$	$\frac{11r}{144}$
Perkel	57	$\{6,5,2;1,1,3\}$	[25],[30]	$\frac{56r}{171}$	$\frac{22r}{57}$	$\frac{68r}{171}$
Mathon(Cycl$(11,5)$)	60	$\{11,8,1;1,2,11\}$	[28]	$\frac{59r}{330}$	$\frac{32r}{165}$	$\frac{r}{5}$
Mathon(Cycl$(19,3)$)	60	$\{19,12,1;1,6,19\}$	[28]	$\frac{59r}{570}$	$\frac{187r}{1710}$	$\frac{3151r}{570}$
Taylor($SRG(29,14)$)	60	$\{29,14,1;1,14,29\}$	[22]	$\frac{59r}{870}$	$\frac{214r}{3045}$	$\frac{29r}{406}$
$GH(2,2)$	63	$\{6,4,4;1,1,3\}$	[29]	$\frac{62r}{189}$	$\frac{76r}{189}$	$\frac{271r}{504}$
$H(3,4)$,Doob	64	$\{9,6,3;1,2,3\}$	[22]	$\frac{7r}{32}$	$\frac{r}{4}$	$\frac{25r}{96}$
Locally Petersen	65	$\{10,6,4;1,2,5\}$		$\frac{64r}{325}$	$\frac{73r}{325}$	$\frac{49r}{156}$
Doro	68	$\{12,10,3;1,3,8\}$		$\frac{67r}{408}$	$\frac{145r}{816}$	$\frac{253r}{1020}$
Doubled Odd(4)	70	{4,3,3,2,2,1,1; 1,1,2,2,3,3,4}	[22]	$\frac{69r}{140}$	$\frac{68r}{105}$	$\frac{869r}{1260}$
$J(8,4)$	70	$\{16,9,4,1;1,4,9,16\}$	[22]	$\frac{69r}{560}$	$\frac{337r}{2520}$	$\frac{691r}{5040}$

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